INVERSE SAMPLING AND OTHER SELECTION PROCEDURES FOR TOURNAMENTS WITH 2 OR 3 PLAYERS

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Milton Sobel* and George H. Weiss+

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+ On leave from National Institutes of Health, Bethesda, Maryland.

1. Introduction

The problem of selecting the best of $k = 2$ or 3 players in a tournament from a ranking and selection point of view is considered when only independent win or lose binary comparisons (or games) are allowed. Some earlier literature on this subject is described in David's monograph [5]. Our main emphasis is on a "Drop the Loser" (DL) sampling rule: For three players, this rule states that the loser of any game is the one that does not play in the very next game. This DL-sampling rule is regarded as a 3-player binary-comparisons analogue of the Play-the-Winner (PW) rule that has been considered by Robbins [9], [10], Isbell [5] and Smith and Pyle [11] for the 2-armed bandit problem and by Zelen [12] for the same problem in the context of clinical trials. We consider some inverse sampling rules with and without early elimination of non-contenders and attempt to compare our results with those of other new sequential procedures.

Roughly speaking, under our formulation we want a procedure R which has a high probability of a correct selection $P\{CS; R\}$ when one of the players is sufficiently better than the others. Let p_1, p_2, and p_3 denote the single-game probability that A beats B, B beats C, and C beats A, respectively, and let $q_i = 1 - p_i$ $(i = 1, 2, 3)$. Let p_0 (which we also write as p below) and P^* denote specified constants with $\frac{1}{2} < p_0 < 1$ and $1/3 < P^* < 1$. We define A to be the best player if and only if $\min(p_1, q_3) > \frac{1}{2}$. Although it is possible under this definition that none of the players is best, we assume that there are only 3 possible decisions available to us, i.e., one of the players has to be selected and declared the best player. We would like to have a procedure R such that for any specified $p_0 > \frac{1}{2}$ and any specified $P^* < 1$

$$(1.1) \quad P\{CS;\ R\} = P\{\text{selecting } A;\ R\} \geq P^* \quad \text{whenever} \quad \min(p_1,\ q_3) \geq p_0,$$

regardless of the value of p_2; here p_2 is regarded as a nuisance parameter and below it is also denoted by θ.

Some of our procedures (see R_I, R_E, and R_S below) do not satisfy (1.1) and we need a weaker form of (1.1). Suppose we only want (1.1) to hold for specific values of θ and for $p_0 > \underline{p} > \frac{1}{2}$, where $\underline{p} = \underline{p}(\theta)$ may depend on θ and on the procedure R. The function $\underline{p}(\theta)$ is given explicitly in (3.9) for procedure R_I; the same expression holds for procedures R_E and R_S also.

Taking A as the best player with $\min(p_1,\ q_3) > p_0$ and θ fixed, we define a least favorable LF_θ configuration with θ fixed to be one in with $p_1 = q_3 = p_0$. For procedure R_I a proof is given by N. Elliott and Y. S. Lin in Appendix B that the configuration $(p_1 = q_3 = p_0)$ is least favorable for fixed θ in the usual sense of minimizing the $P\{CS;\ R_I\}$ over all triples $(p_1,\ p_2,\ p_3)$ with $\min(p_1,\ q_3) \geq p_0$ and fixed $p_2 = \theta$.

For the procedures studied we shall also be concerned with the expected number of games required to reach a decision as a function of p, P^*, and θ; we denote this by $E\{N|LF_\theta\}$.

Since we do not assume the Bradley-Terry Model [3] it is possible to have A preferred to B, B preferred to C, and C preferred to A (i.e., $p_1 > \frac{1}{2}$, $p_2 > \frac{1}{2}$, and $p_3 > \frac{1}{2}$) but such triads will not be in the zone of preference for selection for any of the three players.

In the present discussion we will not consider treating the 3-player problem as a succession of 2-player problems; that is, we do not consider testing player A against B until one is found to be better and then testing C against the better of A and B. The present paper is not meant to be an exhaustive study of the 3-player problem or a search for an optimal procedure. Rather we have analyzed several new procedures (the simpler ones being generally less efficient) and made some efficiency comparisons.

In Section 2 we summarize results for two different 2-player procedures. In Section 3 we analyze the inverse sampling procedure R_I for 3 players (without early elimination of non-contenders) in which the first player to win r games is selected as the winner of the tournament. Exact recurrence relations are derived for the $P\{CS; R_I\}$ and $E\{N; R_I\}$, conditional on the number of wins already obtained by each player. This Markovian character is then exploited to obtain exact numerical values for the $P\{CS|LF_\theta\}$ and $E\{N|LF_\theta\}$ for different values of p, θ, and r; the Bradley-Terry model is represented by the special case $\theta = \frac{1}{2}$.

2. The 2-Player Problem

In the case of 2 contestants all of our procedures can be analyzed exactly and explicit expressions can be derived for the $P\{CS\}$ and $E\{N\}$ functions; we give these results without derivation. In the case of 2 players, one of our procedures (R_I) occurs in [4] and another procedure (R_S) occurs in the work of K. Alam [1]; both of these papers deal with the analogous binomial and multinomial ranking problems. Special cases of our formulas and a discussion of these same two rules R_I

and R_S for the 2-player problem are also given by Kemeny and Snell [7 - p. 165].

Let $p > \frac{1}{2}$ denote the single-game probability that A beats B, so that A denotes the better player. We consider three procedures for selecting the better player.

(1) The first is a single-stage procedure R_F in which an odd number $n = 2m + 1$ of games are played and the winner of a majority of the games is selected as the better player.

(2) The second is an inverse sampling procedure R_I in which the first player to win r games is selected as the better player.

(3) The third is a sequential procedure R_S in which the first player to win d more games than his opponent is selected as the better player.

For the single-stage procedure R_F, the probability of a correct selection is clearly

$$(2.1) \qquad P\{CS; R_F\} = \sum_{j=m+1}^{2m+1} \binom{2m+1}{j} p^j q^{2m+1-j} = I_p(m+1, m+1),$$

where $I_p(a, b)$ is the usual incomplete beta function. These tournaments need only be continued until one player wins $m + 1$ games and then it becomes identical with inverse sampling if we equate r and $m + 1$.

Under the inverse sampling procedure R_I we have

$$(2.2) \qquad P\{CS; R_I\} = p^r \sum_{j=0}^{r-1} \binom{r-1+j}{j} q^j = I_p(r, r)$$

since, for a correct selection, A wins r games and B wins $j < r$ games. The expected number of games in a tournament under inverse sampling is

(2.3) $$E\{N; R_I\} = \sum_{j=0}^{r-1} (j + r)\binom{r+j-1}{j}(p^r q^j + p^j q^r)$$

$$= \frac{r}{p} I_p(r+1, r) + \frac{r}{q} I_q(r+1, r).$$

The expected number W_L of games won by the loser of the tournament is

(2.4) $$E\{W_L; R_I\} = \sum_{j=0}^{r-1} j\binom{r+j-1}{j}(p^r q^j + p^j q^r) = E\{N; R_I\} - r.$$

The sequential procedure R_S can be regarded as a gambler's ruin problem and we easily obtain for $\psi = p/q$

(2.5) $$P\{CS; R_S\} = \frac{\psi^d}{1+\psi^d} .$$

The expected number of games is given by

(2.6) $$E\{N; R_S\} = d\left(\frac{\psi^d-1}{\psi^d+1}\right)\left(\frac{\psi+1}{\psi-1}\right)$$

for $\psi \neq 1$ and equals d^2 for $\psi = 1$. The expected number of games W_L won by the loser is given by

(2.7) $$E\{W_L; R_S\} = \frac{d}{2} \left\{\left(\frac{\psi^d-1}{\psi^d+1}\right)\left(\frac{\psi+1}{\psi-1}\right) - 1\right\}$$

for $\psi \neq 1$ and equals $d(d-1)/2$ for $\psi = 1$.

Table 1A contains a summary of analytically-obtained results for the 2-player inverse sampling procedure R_I giving the value of $r = m + 1$ as a function of p and P^* for $P^* = .75$, .90, .95, and .99. Table 1B contains comparable values of d and $E\{N; R_S\}$ for the sequential procedure R_S. The d-value in the table is actually the smallest integer $\geq d_0$ where d_0 is the solution obtained by setting (2.5) equal to P^*, i.e.

$$d_0 = \{\ln(\frac{P^*}{1-P^*})\}/\ln \psi; \tag{2.8}$$

as a result of using the smallest integer $\geq d_0$, the tabulated values of $E\{N; R_S\}$ are not smooth.

3. Inverse Sampling Procedure R_I for 3 Players.

The procedure R_I consists of a sampling rule, a stopping rule and a decision rule; it is convenient to combine the latter two.

Sampling Rule of Procedure R_I: At the outset randomize between the three possible games (A v. B, B v. C, and C v. A), giving probability 1/3 to each. To determine who plays in succeeding games we use the DL-sampling rule, i.e., the loser of any game sits out the next game.

Stopping and Decision Rule of Procedure R_I: Stop as soon as any one player has a total of r wins and select him as the best player.

Probability of a Correct Selection $P\{CS; R_I\}$.

Let w_A denote the number of games won by A and let $w_A' = r - w_A$; we define w_B , w_B', w_C, and w_C' similarly and use $\underset{\sim}{W}'$ to denote the vector (w_A', w_B', w_C'). Let "wt" denote "wins the tournament" and let "PNG=" denote "the players of the next game are." Let

$$\begin{aligned} R_{k,m,n} &= P\{A \text{ wt} | \underset{\sim}{W}' = (k, m, n) \text{ and PNG} = A \text{ v } B\} \\ S_{k,m,n} &= P\{A \text{ wt} | \underset{\sim}{W}' = (k, m, n) \text{ and PNG} = B \text{ v } C\} \\ T_{k,m,n} &= P\{A \text{ wt} | \underset{\sim}{W}' = (k, m, n) \text{ and PNG} = C \text{ v } A\}, \end{aligned} \tag{3.1}$$

where the procedure R_I is understood. From the DL-sampling rule we obtain the recursive relations

$$R_{k,m,n} = p_1 T_{k-1,m,n} + q_1 S_{k,m-1,n} ,$$

$$(3.2) \qquad S_{k,m,n} = p_2 R_{k,m-1,n} + q_2 T_{k,m,n-1} ,$$

$$T_{k,m,n} = p_3 S_{k,m,n-1} + q_3 R_{k-1,m,n} ,$$

and from the stopping rule we obtain the boundary conditions: for all psoitive k, m, n

$$(3.3) \qquad R_{0,m,n} = T_{0,m,n} = 1$$

$$R_{k,0,n} = S_{k,0,n} = S_{k,m,0} = T_{k,m,0} = 0$$

We have solved (3.2) numerically and calculated $A_{r,r,r}$ defined by

$$(3.4) \qquad A_{r,r,r} = \frac{1}{3}(R_{r,r,r} + S_{r,r,r} + T_{r,r,r}),$$

which gives the exact $P\{CS; R_I\}$ for procedure R_I. This was done for the $LF(\theta)$ configuration, i.e.,

$$(3.5) \qquad \begin{array}{ll} p_1 = q_3 = p \ \text{(say)} & \frac{1}{2} \leq p \leq 1 \\ p_2 = \theta & \frac{1}{2} \leq \theta \leq 1, \end{array}$$

for $r = 1(1) \min(40, r^*)$, where r^* is the smallest value of r for which $P\{CS; R_I\} \geq .99$.

Expected Number of Comparisons $E\{N; R_I\}$.

For $E\{N; R_I\}$ we use a technique similar to the above. Let

$$F_{k,m,n} = E\{N | \underset{\sim}{W}' = (k, m, n) \text{ and } PNG = A \vee B\} ,$$

$$(3.6) \qquad G_{k,m,n} = E\{N | \underset{\sim}{W}' = (k, m, n) \text{ and } PNG = B \vee C\} ,$$

$$H_{k,m,n} = E\{N | \underset{\sim}{W}' = (k, m, n) \text{ and } PNG = C \vee A\} .$$

From the DL-sampling rule we have

$$
\begin{aligned}
F_{k,m,n} &= p_1 H_{k-1,m,n} + q_1 G_{k,m-1,n} + 1 \quad , \\
G_{k,m,n} &= p_2 F_{k,m-1,n} + q_2 H_{k,m,n-1} + 1 \quad , \qquad (3.7) \\
H_{k,m,n} &= p_3 G_{k,m,n-1} + q_3 F_{k-1,m,n} + 1 \quad ,
\end{aligned}
$$

with boundary conditions $F_{k,m,n} = G_{k,m,n} = H_{k,m,n} = 0$ if any one index is zero. Table 4 gives the value of r and $E\{N;R_I\}$ for configurations corresponding to $P^* = .75$, $.90$, $.95$, and $.99$, where $E\{N; R_I\}$ is calculated by

$$(3.8) \qquad E\{N; R_I\} = \frac{1}{3}(F_{r,r,r} + G_{r,r,r} + H_{r,r,r}).$$

Limitations of the Procedure R_I.

Due to the fact that the procedure R_I depends only on the total number of wins for each player an inefficiency arises which is affected by the sampling rule used. For certain points in the parameter space where A is the best player, the $P\{CS\}$, under the sampling rule of Procedure R_I, does not approach 1 as the number of games grows indefinitely. For procedure R_I we delineate these parameter points explicitly. Since our result depends only on the sampling rule and the statistic used (the total number of wins for each player), the same result also holds for procedures R_E and R_S defined below. For the configuration (3.5), our result states that under the DL-sampling rule for the $P\{CS\}$ to approach 1 we need

$$(3.9) \qquad p > \tfrac{1}{2}(-\theta + \sqrt{\theta^2 + 4\theta})$$

and under the vector-at-a-time sampling rule, which gives each player an equal number of games with each of the other players, we need

$$p > \frac{1}{3}(1 + \theta). \tag{3.10}$$

We note that the line in (3.10) is tangent to the curve in (3.9) at $\theta = \frac{1}{2}$ and hence any pair (p, θ) satisfying (3.10) also satisfies (3.9); this is an indication of the improvement obtained by using the DL-sampling rule.

To derive (3.1) consider the first m games for large m and let f_{AB} denote the proportion in which we find A v B playing; define f_{BC} and f_{CA} similarly. Then under the DL-sampling rule

$$\begin{aligned} f_{AB} &= f_{BC}\,\theta + f_{AC}\,q_3 \\ f_{BC} &= f_{AB}\,q_1 + f_{AC}\,p_3\,, \end{aligned} \tag{3.11}$$

and hence

$$f_{AB}(1 - \theta q_1) = f_{AC}(q_3 + \theta p_3). \tag{3.12}$$

After a large number of games the proportion in which we find A playing is given by the two expressions

$$f_{AB} + f_{AC} = f_{AB}\,p_1 + f_{AC}\,q_3 + f_{BC}. \tag{3.13}$$

From (3.13) and the identity $f_{BC} = 1 - f_{AB} - f_{AC}$, we obtain

$$f_{AB}(2 - p_1) + f_{AC}(2 - q_3) = 1. \tag{3.14}$$

Solving (3.12) and (3.14) gives

$$f_{AB} = (q_3 + p_3\theta)/D, \; f_{AC} = (1 - q_1\theta)/D \tag{3.15}$$

where $D = 2 + q_1q_3 + \theta(p_3 - q_1)$. It follows that the long-term frequencies of wins for A, B and C, resp., under the DL-sampling rule are

$$
\begin{aligned}
f_A &= \frac{q_3(1+p_1)+\theta(p_3-q_1)}{D} = \frac{p(1+p)}{2+pq} \\
f_B &= \frac{q_1q_3+\theta(q_1+p_3)}{D} = \frac{q(p+2\theta)}{2+pq} \qquad (3.16)\\
f_C &= \frac{p_1p_3+(1-\theta)(q_1+p_3)}{D} = \frac{q(p+2-2\theta)}{2+pq},
\end{aligned}
$$

where the last expressions in (3.16) are for configuration (3.5). The condition that $f_A > \mathrm{Max}(f_B, f_C)$ is that

$$p_1q_3 > \mathrm{Max}\{q_1\theta, p_3(1-\theta)\}. \qquad (3.17)$$

For configuration (3.5) this reduces to $p^2 > q\theta$, which easily reduces to (3.9).

A corresponding analysis for Vector-at-a-time sampling states that after m vectors for the $P\{CS\}$ to approach 1 as $m \to \infty$ we need

$$f_A = p_1 + q_3 > \mathrm{Max}(f_B, f_C) = \mathrm{Max}\{\theta + q_1, 1-\theta+p_3\}. \qquad (3.18)$$

For the configuration (3.5) this reduces to (3.10).

We note that for the right members of (3.9) and (3.10) are both increasing in θ and equal $(\sqrt{5}-1)/2 = .618\ldots$ and $2/3$ respectively for $\theta = 1$. This indicates that for $p > (\sqrt{5}-1)/2$ under the DL-sampling rule the $P\{CS\} \to 1$ regardless of how close θ is to 1 and a similar statement holds for $p > 2/3$ under the Vector-at-a-time sampling rule.

4. Bounds and Approximations for Inverse Sampling.

In this section an upper bound and approximation for both $P\{CS|LF\}$ and $E\{N|LF\}$ are derived for procedure R_I. Lower bounds for the $P\{CS|LF\}$ can also be found but these do not appear to be as useful, and will not be discussed.

Our derivation of an upper bound is based on a lemma (or three problems) dealing with the number $M = M(r, n)$ of ways of removing r items from an ordered set of n items $(0 \leq r \leq n)$ so that the resulting $r + 1$ groups thus formed by the items remaining all have a preassigned parity; here empty groups are taken into account and their parity is, of course, even.

Lemma: The number M of combinations of r items which (when removed) result in all $r + 1$ groups being even is given by

$$(4.1) \qquad M = \binom{\frac{n+r}{2}}{r} \quad \text{if } \frac{n+r}{2} \text{ is an integer,}$$

and $M = 0$ otherwise. In the above, if we specify that the r^{th} item removed must be the last of the n ordered items, then

$$(4.2) \qquad M = \binom{\frac{n+r}{2} - 1}{r - 1} \quad \text{if } \frac{n+r}{2} \text{ is an integer,}$$

and $M = 0$ otherwise. In addition to the latter condition, if we want the first group to be odd and all the others even, then

$$(4.3) \qquad M = \binom{\frac{n+r-3}{2}}{r - 1} \quad \text{if } \frac{n+r-3}{2} \text{ is an integer,}$$

and $M = 0$ otherwise.

Proof: In the first problem let $x_i \geq 0$ $(i = 1, 2, \ldots, r+1)$ denote the even group sizes, so that $x_1 + x_2 + \ldots + x_{r+1} = n - r$. Setting $x_i = 2y_i$, we obtain the equivalent problem of finding non-negative integer solutions of $y_1 + y_2 + \ldots + y_{r+1} = (n-r)/2$. Setting $z_i = y_i + 1$ gives another equivalent problem of finding positive integer solutions of $z_1 + z_2 + \ldots + z_{r+1} = r + 1 + (n-r)/2$. Hence the answer to our first problem is simply the number of ways of selecting r different spaces from the interior spaces formed by $1 + (n+r)/2$ ordered items, i.e., the combinatorial in (4.1). If $(n+r)/2$ is not an integer there are no solutions.

In the second problem we use the same argument except that there are only r quantities x_i. In the third problem we set $x_1 = 2y_1 + 1$ and $x_i = 2y_i$ for $i > 1$; the remainder of the proof is the same. This proves the lemma.

We now use the symbol A to denote the best of the three players and we let n denote the number of comparisons needed under the DL sampling rule for A to obtain r wins. Let $P_n(A)$ denote the probability that A obtains r wins on precisely the n^{th} game given that A plays in the first game and the DL sampling rule is used; let $P_n(\bar{A})$ denote the same except that A does not play in the first game. Of course, waiting for A (the best player) to obtain r wins is not a procedure; we refer to it as a 'pseudo-procedure'.

<u>PCS Bounds</u>: From the second problem of the lemma (under the LF configuration)

$$(4.4) \qquad P_n(A) = \binom{\frac{n+r}{2} - 1}{r - 1} p^r q^{(n-r)/2}$$

and the sum over n-values $(r \leq n < \infty)$ is one for all $p > 0$. Since R_I can yield wrong results, but the above 'pseudo-procedure' does not, we find that

$$(4.5) \qquad P\{CS|A\} \leq \sum_{n=r}^{3r-2} P_n(A) = p^r \sum_{j=0}^{r-1} \binom{j+r-1}{r-1} q^j = I_p(r, r)$$

where $P\{CS|A\}$ is the $P\{CS|LF\}$ given that A plays in the first game. Similarly using the third problem of the lemma

$$(4.6) \qquad P\{CS|\bar{A}\} \leq \sum_{n=r}^{3r-2} P_n(\bar{A}) = p^r \sum_{j=0}^{r-2} \binom{j+r-1}{r-1} q^j = I_p(r, r-1).$$

From (4.5) and (4.6) we obtain the upper bound UB_1

$$(4.7) \qquad P\{CS|LF\} = \frac{2P\{CS|A\} + P\{CS|\bar{A}\}}{3} \leq \frac{2I_p(r, r) + I_p(r, r-1)}{3} = UB_1.$$

It is easy to show that $I_p(r, r)$ is an upper bound for UB_1 and hence also for the $P\{CS|LF\}$; we denote it by UB_0.

Suppose we get a wrong decision and (say) B wins the tournament. Consider only the games that B wins, arranged in order of occurence. If B beats C twice in succession (in this subset of B's wins), then between these wins A must have lost to C, thus providing one factor q for this event. We get another factor q for each game that A loses to B. Since B wins r games, the minimum power of q is obtained (for example) by alternating the wins of B against C and against A. This gives a minimum power of q equal to [r/2], i.e., for $q \to 0$

$$(4.8) \qquad P\{CS|LF\} \approx \frac{2I_p(r, r) + I_p(r, r-1)}{3} + \mathcal{O}(q^{[r/2]}),$$

where [x] denotes the smallest integer $\geq$ x.

<u>ASN Bounds</u>: Using the lemma again and the n defined above for the 'pseudo-procedure' we obtain upper bounds for $E\{N|LF\}$, where N is the number of games required by procedure R_I. Since n is an upper bound on N, we find that if we start with A

$$(4.9) \qquad E\{N|A\} \leq p^r \sum_{n=r}^{\infty} n \binom{\frac{n+r}{2} - 1}{r - 1} q^{(n-r)/2} = \frac{r(1+q)}{p}$$

and if we start without A the result is increased by exactly 1. Hence

(4.10) $$E\{N|LF\} \leq \frac{r(1+q)}{p} + \frac{1}{3}.$$

A slight improvement on (4.10) is obtained by using the curtailed distribution of n, i.e., by concentrating at $n = 3r - 2$ all the probability for $n \geq 3r - 2$. We give this upper bound

(4.11) $$E\{N|LF\} \leq 3r - 2 - \frac{4}{3}\{(r-1)I_p(r, r) - \frac{qr}{p} I_p(r+1, r-1)\}$$
$$- \frac{1}{3}\{(2r-3)I_p(r, r-1) - \frac{2rq}{p} I_p(r+1, r-2)\}$$

without derivation since the improvement on (4.10) is usually quite small in most cases of interest.

<u>An Improved Upper Bound for the $P\{CS|LF\}$</u>: We now use the fact that, even if we condition on the best player (say, A) obtaining his r^{th} win on exactly the j^{th} trial, half of the games that A does not win, i.e., $[\frac{j-r+1}{2}]$ games, are independent binomial contests between B and C with single-game probabilities of winning θ and $1 - \theta$, respectively, under the LF configuration (3.5).

An improved upper bound for the $P\{CS|LF\}$ is then obtained by distributing A's losses equally between B and C giving the extra one to (say) B when the number is odd. Starting with A as one of the players and using (4.2), we let $i = (j-r)/2$ and obtain for $r \geq 3$

(4.12) $$P\{CS|A\} \leq p^r \sum_{i=0}^{r-1} \binom{i+r-1}{r-1} q^i \sum_{\alpha=[\frac{3i+3}{2}]-r}^{r-1-[\frac{i}{2}]} \binom{i}{\alpha}\theta^{\alpha}(1-\theta)^{i-\alpha}$$

$$= I_p(r, r) - p^r \sum_{i=[\frac{2r+2}{3}]}^{r-1} \binom{i+r-1}{r-1} q^i$$

$$\times \{I_\theta([\frac{2r-i+1}{2}], [\frac{3i-2r+2}{2}]) + I_{1-\theta}([\frac{2r-i}{2}], [\frac{3i-2r+3}{2}])\}.$$

Similarly, starting without A and using (4.3), we obtain for $r \geq 3$

$$(4.13) \qquad P\{CS|\bar{A}\} \leq I_p(r, r-1) - p^r \sum_{i=[\frac{2r}{3}]}^{r-2} \binom{i+r-1}{r-1} q^i$$

$$\times \{I_\theta([\frac{2r-i+1}{2}], [\frac{3i-2r+4}{2}]) + I_{1-\theta}([\frac{2r-i}{2}], [\frac{3i-2r+5}{2}])\},$$

where the sum in (4.13) vanishes for $r = 3$. Combining (4.12) and (4.13) we obtain the desired bound

$$(4.14) \qquad P\{CS|LF\} \leq UB_1 - \frac{p^r}{3}(2\Sigma_1 + \Sigma_2) = UB_2,$$

where Σ_1 and Σ_2 are the sums in (4.12) and (4.13), respectively. For $\theta = 1$ this reduces to

$$(4.15) \qquad P\{CS|LF, \theta = 1\} \leq \frac{1}{3}\{2I_p(r, [\frac{2r+2}{3}]) + I_p(r, [\frac{2r}{3}])\}.$$

Approximations for the P{CS} and ASN: From the large-sample analysis of Section 3 we also obtain another approximation for the PCS which has the same region (see (3.9)) of convergence to one for $r \to \infty$ as the exact P{CS} under the DL-rule. We make use of an identity derived in [8] and utilized in [4] between a sum of multinomial probabilities and a Dirichlet integral; the final result is adjusted because successive observations in the limiting multinomial are not independent under the DL-rule. This derivation has some intrinsic interest since it gives a method for applying the results of independent sampling to the corresponding problem with correlated sampling.

The marginal distribution of the vector (of zeroes and a single one) which shows the player that won the j^{th} game for j large is that of a 3-cell multinomial with the 3 cell probabilities given by (3.16).

Suppose first that the observations are all mutually independent and let s denote the number of such multinomial observations needed for a corresponding P^*-condition; later we make an adjustment for the lack of independence between successive observations under the DL-rule. If we had independent multinomial vector observations then we can use the identity in Theorem 2.4 of [8] (see also (4.3) of [4]) with $k = 3$ and $N = s$ and the $P\{CS\}$ would then be given by the Dirichlet integral

$$(4.16)\qquad I_{b,c}(s,\ s;\ 3s) = \frac{\Gamma(3s)}{\Gamma^3(s)} \int_b^\infty \int_c^\infty \frac{x^{s-1} y^{s-1}}{(1+x+y)^{3s}}\, dxdy$$

where $b = P(W_B)/P(W_A) = q(p+2\theta)/p(1+p)$ and $c = P(W_C)/P(W_A) = q(p+2-2\theta)/p(1+p)$. Integrating by parts in (4.16), we note that two arguments of the I-function get lowered and by iteration this easily leads to

$$(4.17)\qquad I_{b,c}(s,\ s;\ 3s) = \left(\frac{1}{1+b}\right)^s \sum_{j=0}^{s-1} \binom{s-1+j}{s-1} \left(\frac{b}{1+b}\right)^j\ I_{\frac{1+b}{1+b+c}}(s+j,\ s).$$

If we divide by the sum of the coefficients of the I functions in (4.17), then we can treat the sum in (4.17) as an average of incomplete beta functions. If we use the identity found in (4.5), let "$\sim$" denote asymptotic equality in the limit $s \to \infty$ and set $j^{(i)} = j(j-1)\ldots(j-i+1)$, it is easy to show that for $i \geq 0$

$$(4.18)\qquad \left(\frac{1}{1+b}\right)^s \sum_{j=0}^{s-1} j^{(i)} \binom{s-1+j}{s-1} \left(\frac{b}{1+b}\right)^j = s^{(i)} b^i I_{\frac{1}{1+b}}(s+i,\ s-i) \sim s^{(i)} b^i.$$

From this it follows for $s \to \infty$ that the variance of j/s goes to zero, hence we can set j/s equal to b (or j equal to sb) in the I-functions in (4.17). This gives us the desired asymptotic result

$$(4.19) \qquad I_{b,c}(s,\ s;\ 3s) \sim I_{\frac{1}{1+b}}(s,\ s) \quad I_{\frac{1+b}{1+b+c}}(s(1+b),\ s).$$

The first and second I-function, respectively, on the RHS of (4.19) converge to 1 if and only if $p^2 > q\theta$ as in (3.9) and if and only if $p^2 > q(1-\theta)$, where the latter is implied by the former for $\theta \geq 1/2$. In fact for θ close to 1 we can neglect the second I-function since it is much closer to 1 than the first.

We set the right side of (4.19) equal to a specified $P^*(\frac{1}{3} < P^* < 1)$ and solve for s (not necessarily an integer). This constitutes an approximate solution for the problem in [4] with independent sampling.

We now consider the question of adjusting this s-value to obtain the r-value required by the DL-rule where successive observations are correlated. We associate with the $P\{CS\}$ under procedure R_I the statistic $S = \sum_{i=1}^{r} X_i/r$ where $X_i = 1$ if A wins the i^{th} game and 0 if A loses or does not play in the i^{th} game. We want the statistic S to have the same precision (or variance) under the DL-rule as under independent sampling. Using (3.16) we find that in the asymptotic multinomial

$$(4.20) \qquad \rho_1 = \rho(X_i,\ X_{i+1}) = \frac{\frac{p^2(1+p)}{2+pq} - \left(\frac{p(1+p)}{2+pq}\right)^2}{2pq(1+p)^2/(2+pq)^2} = \frac{p}{2}\ .$$

Similarly we can show for each j that in the asymptotic multinomial

$$(4.21) \qquad \rho_j = \rho(X_i,\ X_{i+j}) = (-q)^{j-1}\,\frac{p}{2}\ .$$

A general proof of (4.21) can be obtained by using the second result in the lemma proved above, and the identity

$$p^j \sum_{\alpha=0}^{[\frac{j}{2}]} \binom{j-\alpha}{\alpha}\left(\frac{q}{p^2}\right)^\alpha = \frac{1-(-q)^{j+1}}{1+q} \qquad (j = 0, 1, 2, \ldots) \tag{4.22}$$

proved in Appendix A. Hence the variance of S (if the common variance of each X_i is σ^2) is asymptotically $(r \to \infty)$ given by

$$\begin{aligned} \sigma^2(S) &= \frac{\sigma^2}{r^2}\{r + 2(r-1)\rho_1 + 2(r-2)\rho_2 + \ldots + 2\rho_{r-1}\} \\ &= \frac{\sigma^2}{r^2}\{r + (r-1)p-(r-2)pq + \ldots + p(-q)^{r-2}\} \\ &= \frac{\sigma^2}{r^2}\left(r + \frac{rp}{1+q} - p\left\{\frac{1-(pq)^r}{(1+q)^2}\right\}\right) \sim \frac{2\sigma^2}{r(1+q)} . \end{aligned} \tag{4.23}$$

To attain the same precision we set the last expression equal to σ^2/s obtaining the desired adjustment

$$r = \frac{2s}{1+q} , \tag{4.24}$$

where s is the solution obtained by setting the right side of (4.19) equal to P^*. In this case, we would take the nearest integer to the r-value obtained by (4.24); some numerical values based on (4.19), (4.24) are given in Table 2.

Examination of the computer output shows that $E\{N|LF\}$ is very closely approximated by

$$E\{N|LF\} = Cr + D. \tag{4.25}$$

Values of C and D are given in Table 3 for $.60 \le p \le .85$ for appropriate values of θ. For $p > .85$, C is very close to the upper bound value $(1+q)/p$ and D is very close to $1/3$. In all cases (4.25) differs from the exact value by no more than 1.

TABLE 2: Numerical Illustrations of Approximations, Exact Values and Bounds on r for $p = 0.75$. (Procedure R_I)

	$P^* = 0.90$			$P^* = 0.95$		
	$\theta = 0.50$	$\theta = 0.75$	$\theta = 1.00$	$\theta = 0.50$	$\theta = 0.75$	$\theta = 1.00$
Approximation Based on (4.19),(4.24)	8	9	13	11	13	22
Exact Values	7	9	14	10	13	23
Lower Bounds on r						
Based on (4.14)	7	7	11	8	10	17
Based on (4.7)	5	5	5	6	6	6

TABLE 3: Values of C and D for use in (4.25). (Procedure R_I)

p	θ	C	D
.60	.50	2.40	-3.47
.65	.50	2.13	-1.62
	.75	2.11	-2.19
.70	.50	1.88	- .40
	.75	1.89	- .92
	1.00	1.89	-1.88
.75	.50	1.67	.16
	.75	1.68	- .10
	1.00	1.70	- .70
.80	.50	1.50	.31
	.75	1.50	.24
	1.00	1.51	.04
.85	.50	1.35	.33
	.75	1.35	.33
	1.00	1.34	.29

5. Inverse Sampling with Elimination: Procedure R_E.

For 3 players we develop a Markovian procedure R_E based on inverse sampling and the DL-rule which eliminates one of the three players at or before the tournament is terminated. As in the case of R_I the recurrence formulae provide an algorithm for obtaining exact answers for the $P\{CS|LF\}$ and $E\{N|LF\}$. By building up a table of these functions with $P\{CS|LF\}$-values increasing to one, we can find the specific procedure satisfying any given P^*-condition and the associated $E\{N|LF\}$-value.

At the outset we use randomization and play A v B, B v C or C v A each with probability 1/3. The DL-rule is used for sampling and we eliminate any player as soon as he accumulates a total of r losses. The remaining player is then declared the winner.

When there is no confusion we refer to A as the best player. Let L_A, L_B and L_C denote the number of losses of A, B and C, respectively; let $L'_A = r - L_A$, $L'_B = r - L_B$ and $L'_C = r - L_C$. In analogy with (3.10) we define

$$R_{k,m,n} = P\{A \text{ wins the tournament} | L'_A = k,\ L'_B = m,\ L'_C = n \text{ and the next game is } A \text{ v } B\} \tag{5.1}$$

and similarly $S_{k,m,n}$ for B v C and $T_{k,m,n}$ for C v A. From the DL sampling rule we obtain

$$\begin{aligned} R_{k,m,n} &= p_1 T_{k,m-1,n} + q_1 S_{k-1,m,n} \\ S_{k,m,n} &= p_2 R_{k,m,n-1} + q_2 T_{k,m-1,n} \\ T_{k,m,n} &= p_3 S_{k-1,m,n} + q_3 R_{k,m,n-1} \end{aligned} \tag{5.2}$$

and from the elimination rule we obtain the "boundary" conditions

$$
\begin{aligned}
S_{0,m,n} &= 0, \\
R_{k,m,0} &= V_{k,m} \quad \text{(say)}, \\
T_{k,0,n} &= V_{k,n} \quad \text{(say)}, \qquad (5.3) \\
U_{k,0} &= V_{k,0} = 1, \\
U_{0,m} &= V_{0,n} = 0.
\end{aligned}
$$

After player C is eliminated we use the recurrence formula

$$U_{k,m} = p_1 U_{k,m-1} + q_1 U_{k-1,m} \qquad (5.4)$$

and after player B is eliminated we use

$$V_{k,m} = p_3 V_{k,n-1} + q_3 V_{k-1,n}. \qquad (5.5)$$

After solving these equations we calculated $PCS(r)$ given by

$$PCS(r) = \frac{1}{3}(R_{r,r,r} + S_{r,r,r} + T_{r,r,r}) \qquad (5.6)$$

which is the exact $P\{CS\}$ for the procedure R_E. This was done for the configuration (3.5) for $r = 1(1)\min(40, r^*)$ where r^* is the smallest value of r for which $P\{CS; R_E\} \geq .99$.

For $E\{N\}$ under procedure R_E we define functions $F_{k,m,n}$, $G_{k,m,n}$ and $H_{k,m,n}$ as in (3.6) and write as in (3.7)

$$
\begin{aligned}
F_{k,m,n} &= p_1 H_{k,m-1,n} + q_1 G_{k-1,m,n} + 1 \\
G_{k,m,n} &= p_2 F_{k,m,n-1} + q_2 H_{k,m-1,n} + 1 \qquad (5.7) \\
H_{k,m,n} &= p_3 G_{k-1,m,n} + q_3 F_{k,m,n-1} + 1.
\end{aligned}
$$

The boundary conditions assert that $F_{k,m,n}$, $G_{k,m,n}$ and $H_{k,m,n}$ are equal to 0 if two or more subscripts are zero. Letting $X_{k,m}$ denote the common value of $F_{k,m,0} = G_{k,m,0} = H_{k,m,0}$, $Y_{m,n}$ denote the common value of $F_{0,m,n} = G_{0,m,n} = H_{0,m,n}$ and $Z_{k,n}$ denote the common value of $F_{k,0,n} = G_{k,0,n} = H_{k,0,n}$, we then use

$$\begin{aligned} X_{k,m} &= p_1 X_{k,m-1} + q_1 X_{k-1,m} + 1 \\ Y_{m,n} &= p_2 Y_{m,n-1} + q_2 Y_{m-1,n} + 1 \\ Z_{k,n} &= p_3 Z_{k-1,n} + q_3 Z_{k,n-1} + 1 \end{aligned} \tag{5.8}$$

to complete the algorithm. Solving these equations for the configuration (3.5), we then computed

$$E\{N; R_E\} = \frac{1}{3}(F_{r,r,r} + G_{r,r,r} + H_{r,r,r}) \tag{5.9}$$

which is the exact value of $E\{N|LF\}$ for procedure R_E. Table 4[§] gives the values of r and $E\{N; R_E\}$ for the configuration (3.5) corresponding to selected values of p, θ and P^*.

6. Other Sequential Procedures.

For the purpose of comparison we include a discussion of several sequential procedures with Monte Carlo results. In the first one, called procedure R_S, we used only Monte Carlo methods; in the remaining procedures we make use of some theory in [2] for the P^*-condition and use Monte Carlo methods to estimate the $P\{CS|LF_\theta\}$ and $E\{N|LF_\theta\}$.

Let W_A, W_B and W_C denote the number of games won by A, B and C, respectively; let the ordered values be denoted by $W_1 \leq W_2 \leq W_3$, where ties are clearly possible. For the procedure R_S we use the DL-sampling rule and for $d > 0$ the

§ Table 4 on page 28.

Stopping Rule for Procedure R_S: Terminate as soon as $W_3 - W_2 \geq d$ and select the player with W_3 wins as best.

Monte Carlo results for the LF_θ configuration (3.5) were obtained for $d = 2(2)\,10$ and selected values of p and θ. The proportion of successes (PS), i.e. of correct selections, in 1000 tournaments is used as an estimate of the $P\{CS|LF_\theta\}$. The smallest value of d for which this estimate exceeds P^* is taken as the required minimum value needed to satisfy the P^*-condition (1.1); these are given in Table 4 along with the Monte Carlo estimate $\overline{N}$ of $E\{N|LF_\theta\}$ for selected values of p, θ and P^*.

For the remaining three sequential procedures we use a common notation and make use of the remark on page 14 of [2] that in order for the P^*-condition (see (3.1.11) in [2]) to hold, it is not necessary to take the observations a vector at a time, i.e., to follow the cyclic pattern A v B, B v C, C v A. Thus, for example, we can use the same result for a procedure R_{DL} which uses the DL sampling rule. This result states that if we use a certain likelihood-ratio statistic W (to be defined explicitly below) and stop at any point when $W \geq P^*$ then the P^*-condition will be satisfied. All of the three remaining procedures have the Wald Stopping Structure (see page 17 of [2]) since they stop as soon as $W \geq P^*$ and choose among the decision (or hypotheses) with maximum likelihood. It should be carefully noted that we have only considered the identification aspect of the problem since no monotonicity properties have been shown with respect to p or θ; in other words, if we knew the values of the common p and θ then we can select the best player with the desired value of P^*. For the ranking aspect of this problem we have to show that if the true

$p \geq p_0$ (the specified value) and the known value of θ is used, then the achieved $P\{CS\}$ will be at least as large as P^*; this has not been done. Finally we are interested in the robustness of these procedures with respect to θ, i.e., in the values of $P\{CS|LF_\theta\}$ when θ_1 is used in the statistic W defined below and $\theta \neq \theta_1$ is the true value; this also has not been investigated in this paper.

To introduce W, we first define W_{AB} as the number of games in which A beat B and similarly for W_{BA}, W_{AC}, W_{CA}, W_{BC} and W_{CB}. Six likelihoods are defined for the LF_θ configuration (3.5) by

$$
\begin{aligned}
L_{ABC} &= p^{W_{AB}} q^{W_{BA}} \theta^{W_{BC}} (1-\theta)^{W_{CB}} p^{W_{AC}} q^{W_{CA}} \\
L_{ACB} &= p^{W_{AC}} q^{W_{CA}} \theta^{W_{CB}} (1-\theta)^{W_{BC}} p^{W_{AB}} q^{W_{BA}} \\
L_{BAC} &= p^{W_{BA}} q^{W_{AB}} \theta^{W_{AC}} (1-\theta)^{W_{CA}} p^{W_{BC}} q^{W_{CB}} \\
L_{BCA} &= p^{W_{BC}} q^{W_{CB}} \theta^{W_{CA}} (1-\theta)^{W_{AC}} p^{W_{BA}} q^{W_{AB}} \\
L_{CAB} &= p^{W_{CA}} q^{W_{AC}} \theta^{W_{AB}} (1-\theta)^{W_{BA}} p^{W_{CB}} q^{W_{BC}} \\
L_{CBA} &= p^{W_{CB}} q^{W_{BC}} \theta^{W_{BA}} (1-\theta)^{W_{AB}} p^{W_{CA}} q^{W_{AC}} .
\end{aligned}
\tag{6.1}
$$

The sum of L_{ABC} and L_{ACB} represents the "total likelihood" that the common $p(= p_1 = q_3)$ is associated with player A, i.e., that A is the best player. Hence we further define

$$
\begin{aligned}
\mathcal{L}_A &= L_{ABC} + L_{ACB} \\
\mathcal{L}_B &= L_{BAC} + L_{BCA} \\
\mathcal{L}_C &= L_{CAB} + L_{CBA}
\end{aligned}
\tag{6.2}
$$

and the value of W can then be written as

$$(6.3) \qquad W = \frac{\max(\mathcal{L}_A, \mathcal{L}_B, \mathcal{L}_C)}{\mathcal{L}_A + \mathcal{L}_B + \mathcal{L}_C} .$$

For all the three remaining procedures we use the common

Stopping and Terminal Decision Rule: Stop as soon as $W \geq P^*$ and select as the best player A if $\mathcal{L}_A$ is the maximum (6.3), B if $\mathcal{L}_B$ is the maximum and C if $\mathcal{L}_C$ is the maximum. For $P^* > \frac{1}{2}$ we cannot have ties for first place at termination. For $\frac{1}{3} < P^* \leq \frac{1}{2}$ we may have ties at termination; if t players are tied for first place then we select one by an independent experiment which gives equal probability, i.e., $1/t$, to each of them.

It follows that these three procedures differ only in the sampling rule. Since R_{DL} uses the DL sampling rule, we need only define the sampling rule for the remaining two.

For the "Double Duty" procedure R_{DD} we use the

Sampling Rule: Find the two largest $\mathcal{L}$'s and play the corresponding two players in the next game. In the case of ties and at the outset we use the appropriate randomization, ie.e., we use $\frac{1}{2}$ when two are tied for 1st. or 2nd. place and 1/3 when there is a triple tie.

For procedure R_{DDR} we define

$$(6.4) \qquad \begin{aligned} s_1 &= (\mathcal{L}_A + \mathcal{L}_B)/2(\mathcal{L}_A + \mathcal{L}_B + \mathcal{L}_C), \\ s_2 &= (\mathcal{L}_B + \mathcal{L}_C)/2(\mathcal{L}_A + \mathcal{L}_B + \mathcal{L}_C), \\ s_3 &= (\mathcal{L}_C + \mathcal{L}_A)/2(\mathcal{L}_A + \mathcal{L}_B + \mathcal{L}_C), \end{aligned}$$

which add to one and use the

Sampling Rule: Randomize by taking a uniform deviate U $(0 \leq U < 1)$ and play A v B if $0 \leq U < s_1$, play B v C if $s_1 \leq U < s_1 + s_2$, and play C v A if $s_1 + s_2 \leq U < 1$.

Procedures R_{DD} and R_{DDR} eliminate noncontenders in a more natural way by gradually lowering the probability of putting them into the next game rather than by a sudden complete withdrawal from the tournament.

Evaluation of Monte Carlo Results:

The Monte Carlo PS results based on 300 tournaments in Table 4 for the last three sequential procedures R_{DL}, R_{DDR} and R_{DD} indicate that the nominal P^*-value is satisfied since the observed PS value went below P^* in only 8 of the 252 cases (or 3%) and the maximum error is .017. For large values of p (say .95), all the procedures have their $E\{N|LF_\theta\}$ (or $\bar{N}$ values) approximately equal and there is not much practical basis for using this as a criterion for choosing one of them. For smaller values of p (say, .65 to .75), ; the differences are substantial. If we look at the maximum over the three values of θ studied for each p, then procedure R_{DD} shows a reduction of as much as 50% when compared to the results of procedure R_I for some values of p and P^*. In general, the results for procedure R_{DD} appear to be better, if we look at the maximum over three values of θ, then any of the other procedures considered.

In the absence of any information about θ, it might be desirable to use a closer net of θ-values and find the value of θ which maximizes $\bar{N}$ and treat the problem as if that were the true value of θ. A more practical approach might be to use the accumulated observations to estimate θ but this has not been considered in this paper.

It is also interesting to note that the value of the statistic W (defined in (6.3) above) at termination is another estimate of the $P\{CS|LF_\theta\}$; the theoretical basis for this is discussed in Section 3.2 of [2]. Monte Carlo values of W averaged over 300 experiments are denoted by $\overline{W}$ and given for Procedure R_{DD} in Table 4.

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TABLE 4: 3 Players Problem. Comparison of $E\{N|LF_\theta\}$ values (exact and estimated) for six procedures (the r and d values are needed to make the associated procedure explicit; PS and $\overline{N}$ are Monte Carlo (MC) estimates of the $P\{CS|LF_\theta\}$ and $E\{N|LF_\theta\}$ based on 300 and 600 tournaments, respectively).

$P^* = .75$

p	θ	R_I (exact)		R_E (exact)		R_S (MC)		R_{DL} (MC)		R_{DDR} (MC)		R_{DD} (MC)		
		$E\{N\|LF_\theta\}$	r	$E\{N\|LF_\theta\}$	r	$\overline{N}^{\S}$	$d^{\S}$	$\overline{N}$	PS	$\overline{N}$	PS	$\overline{N}$	PS	$\overline{W}$
.65	.50	17.4	9	18.5	9	15.7	4	17.0	.763	15.1	.760	13.5	.760	.792
	.75	25.3	13	18.4	7	22.5^+	5^+	16.2	.733	16.3	.806	12.5	.817	.780
	1.00	--	#	23.8	9	--	-	10.6	.930	8.1	.883	9.5	.770	.813
.70	.50	8.4	5	10.0	4	8.8^+	3^+	9.0	.813	9.7	.803	8.3	.803	.811
	.75	10.2	6	9.9	4	13.4	4	9.8	.823	10.4	.763	7.4	.777	.793
	1.00	20.8	12	9.9	4	18.4^+	5^+	7.8	.840	6.8	.817	6.4	.840	.801
.75	.50	6.2	4	7.2	3	7.5^+	3^+	6.1	.803	6.6	.820	5.4	.803	.816
	.75	6.1	4	7.2	3	7.6^+	3^+	5.5	.817	6.6	.767	5.7	.850	.813
	1.00	7.6	5	7.1	3	7.6^+	3^+	5.9	.870	5.7	.870	4.9	.860	.830
.80	.50	4.2	3	4.5	2	3.5^+	3^+	2.9	.767	4.0	.820	3.1	.797	.786
	.75	4.2	3	4.5	2	3.7	2	2.8	.753	4.4	.787	3.0	.777	.774
	1.00	4.1	3	4.5	2	3.6	2	2.8	.747	3.7	.830	3.1	.810	.808
.85	.50	2.5	2	2.0	1	3.5	2	2.8	.837	3.5	.893	2.6	.853	.833
	.75	2.5	2	2.0	1	3.4	2	2.7	.817	3.6	.807	2.6	.870	.828
	1.00	2.5	2	2.0	1	3.4	2	2.7	.830	3.2	.810	2.5	.793	.821
.90	.50	2.5	2	2.0	1	3.1	2	2.0	.800	2.4	.823	2.0	.853	.845
	.75	2.5	2	2.0	1	3.3	2	2.0	.833	2.7	.810	2.0	.853	.839
	1.00	2.5	2	2.0	1	3.1	2	2.0	.820	2.8	.830	2.0	.833	.840
.95	.50	2.4	2	2.0	1	3.0	2	2.0	.910	2.3	.887	2.0	.923	.919
	.75	2.4	2	2.0	1	2.9	2	2.0	.913	2.5	.907	2.0	.927	.917
	1.00	2.4	2	2.0	1	2.9	2	2.0	.923	2.8	.930	2.0	.917	.919

\# indicates that the value of r is greater than 40.

\+ indicates values based on linear interpolation in a table of Monte Carlo results.

§ estimates based on 1000 tournaments

TABLE 4: continued

$P^* = .90$

p	θ	R_I (exact)		R_E (exact)		R_S (MC)		R_{DL} (MC)		R_{DDR} (MC)		R_{DD} (MC)		
		$E\{N\|LF_\theta\}$	r	$E\{N\|LF_\theta\}$	r	$\overline{N}^§$	$d^§$	$\overline{N}$	PS	$\overline{N}$	PS	$\overline{N}$	PS	$\overline{W}$
.65	.50	41.1	20	54.3	20	30.8	6	30.6	.937	30.9	.890	26.7	.890	.915
	.75	--	#	76.8	28	54.2^+	9	30.2	.950	30.3	.923	22.9	.923	.913
	1.00	--	#	65.1	24	--	-	12.8	1.000	11.6	.973	16.1	.917	.923
.70	.50	20.2	11	23.3	9	17.9^+	5^+	16.5	.910	16.8	.927	14.4	.920	.921
	.75	29.3	16	28.6	11	23.9	6	17.9	.950	19.0	.943	13.8	.903	.918
	1.00	68.0	37	33.9	13	45.3^+	9^+	11.8	.967	10.2	.953	11.2	.943	.932
.75	.50	11.5	7	14.9	6	11.0	4	9.7	.943	10.8	.920	9.2	.937	.924
	.75	14.9	9	14.6	6	14.8^+	5^+	11.6	.933	12.3	.930	10.2	.963	.933
	1.00	23.0	14	20.0	8	19.0	6	9.5	.970	8.6	.960	8.0	.927	.934
.80	.50	7.5	5	9.5	4	9.5	4	7.1	.943	7.8	.913	6.5	.943	.934
	.75	9.0	6	9.5	4	9.3	4	7.8	.923	8.4	.933	6.7	.927	.937
	1.00	11.9	8	11.9	5	9.8	4	6.3	.953	6.8	.930	5.5	.927	.925
.85	.50	5.5	4	6.8	3	5.8^+	3^+	5.0	.947	5.4	.947	4.7	.960	.933
	.75	5.5	4	6.8	3	5.6^+	3^+	5.1	.933	5.9	.960	4.6	.917	.927
	1.00	6.8	5	9.1	4	5.6^+	3^+	5.5	.963	6.0	.963	4.7	.970	.947
.90	.50	3.8	3	4.3	2	3.1	2	4.2	.957	4.6	.953	3.7	.923	.948
	.75	3.8	3	4.3	2	3.3	2	4.1	.937	4.6	.943	3.7	.957	.941
	1.00	3.8	3	6.6	3	3.1	2	4.2	.953	4.6	.943	3.7	.963	.949
.95	.50	2.4	2	2.0	1	3.0	2	2.5	.940	3.1	.943	2.5	.967	.954
	.75	2.4	2	2.0	1	2.9	2	2.5	.933	3.4	.943	2.6	.953	.953
	1.00	2.4	2	4.2	2	2.9	2	2.7	.957	3.1	.950	2.4	.933	.946

TABLE 4: continued

$P^{*} = .95$

p	θ	R_I (exact)		R_E (exact)		R_S (MC)		R_{DL} (MC)		R_{DDR} (MC)		R_{DD} (MC)		
		$E\{N\|LF_\theta\}$	r	$E\{N\|LF_\theta\}$	r	$\overline{N}^{\S}$	$d^{\S}$	$\overline{N}$	PS	$\overline{N}$	PS	$\overline{N}$	PS	$\overline{W}$
	.50	58.0	28	65.1	24	43.0	8	40.8	.957	38.5	.967	33.8	.973	.959
.65	.75	--	#	84.5	31	--	-	41.9	.963	40.9	.957	30.8	.973	.957
	1.00	--	#	--	#	--	-	12.8	1.000	13.2	.997	21.4	.977	.965
	.50	29.7	16	33.9	13	22.4	6	22.9	.963	21.5	.970	18.4	.957	.963
.70	.75	46.3	25	36.6	14	41.5	10	24.8	.947	24.9	.963	18.6	.953	.959
	1.00	--	#	44.6	17	--	-	12.5	1.000	12.0	.990	13.7	.940	.965
	.50	16.7	10	20.0	8	14.4^{+}	5^{+}	13.1	.963	14.2	.960	11.9	.970	.963
.75	.75	21.7	13	20.0	8	18.4	6	14.9	.963	15.9	.967	12.0	.970	.962
	1.00	38.3	23	22.5	9	27.3	8	10.5	.953	10.0	.977	9.6	.973	.968
	.50	10.6	7	11.9	5	9.5	4	8.6	.973	9.5	.957	8.0	.980	.964
.80	.75	12.1	8	11.9	5	9.3	4	10.3	.970	10.6	.960	8.7	.953	.968
	1.00	18.1	12	11.9	5	15.2	6	8.9	.977	8.6	.983	7.3	.970	.969
	.50	7.0	5	9.5	4	8.0	4	6.3	.973	7.3	.957	5.9	.973	.968
.85	.75	8.3	6	9.5	4	5.6^{+}	3^{+}	6.8	.977	7.7	.980	6.1	.983	.971
	1.00	9.6	7	9.5	4	7.9	4	6.0	.967	6.6	.973	5.2	.957	.967
	.50	5.1	4	6.6	3	4.7^{+}	3^{+}	4.4	.967	5.0	.957	4.4	.977	.969
.90	.75	5.1	4	6.6	3	5.0^{+}	3^{+}	4.4	.973	5.3	.973	4.2	.967	.966
	1.00	5.1	4	6.6	3	4.9^{+}	3^{+}	5.1	.990	6.1	.983	4.6	.983	.983
	.50	3.6	3	4.2	2	3.0	2	3.9	.980	4.0	.980	3.3	.983	.979
.95	.75	3.6	3	4.2	2	2.9	2	3.8	.997	4.1	.983	3.3	.987	.975
	1.00	3.6	3	4.2	2	2.9	2	3.7	.990	4.4	.963	3.5	.963	.978

TABLE 4: continued

$P^* = .99$

p	θ	R_I (exact)		R_E (exact)		R_S (MC)		R_{DL} (MC)		R_{DDR} (MC)		R_{DD} (MC)		
		$E\{N\|LF_\theta\}$	r	$E\{N\|LF_\theta\}$	r	$\bar{N}^§$	$d^§$	$\bar{N}$	PS	$\bar{N}$	PS	$\bar{N}$	PS	$\bar{W}$.
.65	.50	--	#	--	#	--	-	60.6	.997	57.0	.997	49.2	.990	.992
	.75	--	#	--	#	--	-	62.9	.987	59.7	.983	47.9	1.000	.991
	1.00	--	#	--	#	--	-	12.5	1.000	13.4	1.000	30.3	.993	.992
.70	.50	52.3	28	60.0	23	35.3	9	32.8	.990	33.5	.997	28.7	1.000	.992
	.75	--	#	71.1	27	--	-	39.0	.997	38.2	.997	28.4	.993	.992
	1.00	--	#	87.0	33	--	-	13.1	1.000	13.6	1.000	18.7	.990	.993
.75	.50	30.3	18	35.1	14	23.7	8	20.6	.990	19.4	.997	17.6	1.000	.993
	.75	41.8	25	37.6	15	31.6	10	24.4	.993	24.8	.997	18.5	.994	.992
	1.00	--	#	42.8	17	--	-	12.7	1.000	13.1	1.000	13.5	.997	.994
.80	.50	18.3	12	21.6	9	14.2	6	13.2	.990	13.8	.990	11.4	.993	.993
	.75	22.8	15	21.6	9	16.8^+	7^+	15.2	.990	16.1	.990	12.4	.990	.993
	1.00	33.3	22	24.0	10	20.8	8	11.5	.990	11.4	.997	9.7	.997	.994
.85	.50	11.1	8	13.8	6	9.6^+	5^+	9.3	.993	9.9	.990	8.0	.990	.993
	.75	13.8	10	13.8	6	9.8^+	5^+	9.7	.997	10.9	.997	9.0	1.000	.994
	1.00	17.9	13	13.7	6	11.8	6	8.4	.997	9.0	1.000	7.6	.997	.993
.90	.50	7.6	6	8.8	4	6.4	4	6.0	.997	7.0	.993	5.7	.997	.993
	.75	7.6	6	8.8	4	6.7	4	6.2	.983	7.6	.993	5.7	.993	.992
	1.00	10.1	8	8.8	4	6.7	4	6.9	.997	8.0	.997	6.0	.993	.994
.95	.50	4.7	4	6.3	3	5.5	4	4.6	1.000	5.4	.997	4.5	.993	.997
	.75	4.7	4	6.3	3	5.5	4	4.6	.997	5.8	1.000	4.5	.993	.997
	1.00	5.8	5	6.3	3	5.6	4	4.5	.997	5.6	.997	4.3	1.000	.996

APPENDIX A: Proof of the Identity in (4.22).

We consider the sum

$$(A.1) \qquad F_j = \sum_{\alpha=0}^{[\frac{j}{2}]} \binom{j-\alpha}{\alpha} x^\alpha = \sum_{\alpha=0}^{\infty} \binom{j-\alpha}{\alpha} x^\alpha$$

where the sum is extended to ∞, since the binomial coefficients are zero for $\alpha > [j/2]$. We define the generating function

$$(A.2) \qquad F(s) = \sum_{j=0}^{\infty} F_j s^j = \sum_{\alpha=0}^{\infty} x^\alpha \sum_{j=2\alpha}^{\infty} \binom{j-\alpha}{\alpha} s^j$$

$$= \sum_{\alpha=0}^{\infty} (xs^2)^\alpha \sum_{m=0}^{\infty} \binom{\alpha+m}{\alpha} s^m = \sum_{\alpha=0}^{\infty} \frac{(xs^2)^\alpha}{(1-s)^{\alpha+1}}$$

$$= (1 - s - xs^2)^{-1}.$$

But this can be expanded by partial fractions and leads to the final result

$$(A.3) \qquad F_j = \frac{1}{2^{j+1}\sqrt{1+4x}} \{(1 + \sqrt{1+4x})^{j+1} - (1 - \sqrt{1+4x})^{j+1}\} .$$

This reduces to (4.22) when x is set equal to q/p^2.

APPENDIX B

The enclosed Table 5 contains two procedures not previously defined and two that were. The new procedures are defined so that the decrease (or increase) in $E\{N|LF_\theta\}$ for R_I compared to R_{IC} and for R_{DL} compared to R_{BKS} is due solely to the DL-sampling rule.

Procedure R_{IC} is an inverse sampling procedure with a cyclic sampling pattern, i.e., we start by randomizing (with equal probability) between A vs. B, B vs. C and C vs. A and then continue in a cyclic manner with these three types of games (in any prescribed order). Using the same definitions as in (3.1), except that we have a new sampling rule, the recursion formulas become

$$R_{k,\,m,\,n} = p_1 S_{k-1,\,m,\,n} + q_1 S_{k,\,m-1,\,n}$$

$$\text{(B 1)} \qquad S_{k,\,m,\,n} = p_2 T_{k,\,m-1,\,n} + q_2 T_{k,\,m,\,n-1}$$

$$T_{k,\,m,\,n} = p_3 R_{k,\,m,\,n-1} + q_3 R_{k-1,\,m,\,n}.$$

The boundary conditions are the same as in (3.3). [The two essential changes to make (B 1) the exact equations for $E\{N\}$, namely adding 1 to each equation and replacing 1 by 0 in the boundary conditions, are the same as in (3.7) and need not be repeated.] As in Section 3 we are interested in (3.4) and (3.8). Comparable values of $E\{N|LF_\theta\}$ for procedures R_{IC} and R_I are given in Table 5 for selected values of (p, θ, P^*). The procedure R_I shows an improvement over R_{IC} in all cases computed and in several cases the reduction is as much as 50%.

The procedure R_{BKS} samples a vector at a time and hence the observed number of games is a multiple of 3 in every tournament. After each set of 3 games the statistic W in (6.3) is calculated and we stop as soon as

$W \geq P^*$ as in Section 6, the only difference between this procedure and R_{DL} being that the latter uses the DL-sampling rule. Except for six cases, in each of which $\theta = 1$, p is small and P^* is moderate (see starred entries in Table 5), the procedure R_{DL} shows an improvement over R_{BKS}; the reduction is less than that observed in the previous comparison but it still surpasses 25% reduction in many cases.

The proportion of successes (PS) in 300 trials went below P^* in 2 out of 64 cases for R_{BKS}, the maximum difference $(P^* - PS)$ being .003. For R_{DL} the PS-value went below P^* in 5 out of 64 cases, the maximum difference $(P^* - PS)$ being .017. Since these dips below P^* are easily explainable by chance, we have an empirical verification of the basic P^*-condition (1.1) and at the same time a numerical study of the improvement due only to the DL-sampling rule.

TABLE 5: Comparison of Procedures for Evaluating the Improvement Arising Only from the DL-sampling Rule.

		$P^* = .75$					$P^* = .90$				
p	θ	R_{IC}(exact)		R_I(exact)	R_{BKS}(MC)	R_{DL}(MC)	R_{IC}(exact)		R_I(exact)	R_{BKS}(MC)	R_{DL}(MC)
		$E\{N\|LF_\theta\}$	r	$E\{N\|LF_\theta\}$	$\bar{N}$	$\bar{N}$	$E\{N\|LF_\theta\}$	r	$E\{N\|LF_\theta\}$	$\bar{N}$	$\bar{N}$
.65	.50	19.3	9	17.4	18.6	17.0	45.3	20	41.1	39.4	30.6
	.75	41.9	19	25.3	20.2	16.2	---	#	---	36.5	30.2
	1.00	---	#	---	8.7*	10.6*	---	#	---	10.8*	12.8*
.70	.50	11.8	6	8.4	11.8	9.0	25.1	12	20.2	22.9	16.5
	.75	13.8	7	10.2	12.9	9.8	46.3	22	29.3	25.0	17.9
	1.00	---	#	20.8	7.7*	7.8*	---	#	68.0	10.3*	11.8*
.75	.50	7.1	4	6.2	10.8	6.1	15.5	8	11.5	15.0	9.7
	.75	7.1	4	6.1	9.3	5.5	21.4	11	14.9	17.0	11.6
	1.00	18.8	10	7.6	7.5	5.9	69.2	35	23.0	10.9	9.5
.80	.50	4.9	3	4.2	3.7	2.9	8.9	5	7.5	9.9	7.1
	.75	4.9	3	4.2	3.8	2.8	12.6	7	9.0	10.1	7.8
	1.00	6.7	4	4.1	3.6	2.8	23.8	13	11.9	7.4	6.3
.85	.50	2.8	2	2.5	3.4	2.8	6.6	4	5.5	9.2	5.0
	.75	2.8	2	2.5	3.5	2.7	8.4	5	5.5	8.9	5.1
	1.00	2.8	2	2.5	3.5	2.7	11.9	7	6.8	7.4	5.5
.90	.50	2.8	2	2.5	3.3	2.0	4.7	3	3.8	7.7	4.2
	.75	2.8	2	2.5	3.3	2.0	4.6	3	3.8	7.3	4.1
	1.00	2.8	2	2.5	3.3	2.0	6.2	4	3.8	6.9	4.2
.95	.50	2.7	2	2.4	3.1	2.0	2.7	2	2.4	3.1	2.5
	.75	2.7	2	2.4	3.1	2.0	2.7	2	2.4	3.2	2.5
	1.00	2.7	2	2.4	3.2	2.0	2.7	2	2.4	3.1	2.7

* Exceptions to the observed result that the DL-sampling rule reduces $\bar{N}$ occur only for values of θ near 1.

\# See Table 4.

TABLE 5: continued

		$P^* = .95$					$P^* = .99$				
p	θ	R_{IC}(exact) $E\{N\|LF_\theta\}$	r	R_I(exact) $E\{N\|LF_\theta\}$	R_{BKS}(MC) $\overline{N}$	R_{DL}(MC) $\overline{N}$	R_{IC}(exact) $E\{N\|LF_\theta\}$	r	R_I(exact) $E\{N\|LF_\theta\}$	R_{BKS}(MC) $\overline{N}$	R_{DL}(MC) $\overline{N}$
.65	.50	66.3	29	58.0	50.4	40.8	---	#	---	70.7	60.6
	.75	---	#	---	46.0	41.9	---	#	---	67.7	62.9
	1.00	---	#	---	11.7*	12.8*	---	#	---	12.5	12.5
.70	.50	36.0	17	29.7	25.8	22.9	61.8	29	52.3	41.2	32.8
	.75	74.4	35	46.3	33.0	24.8	---	#	---	48.6	39.0
	1.00	---	#	---	11.4*	12.5*	---	#	---	13.2	13.1
.75	.50	21.6	11	16.7	18.6	13.1	37.7	19	30.3	27.2	20.6
	.75	33.5	17	21.7	20.4	14.9	65.7	33	41.8	32.5	24.4
	1.00	---	#	38.3	11.1	10.5	---	#	---	13.6	12.7
.80	.50	14.6	8	10.6	13.6	8.6	24.1	13	18.3	17.4	13.2
	.75	18.4	10	12.1	13.0	10.3	35.3	19	22.8	20.2	15.2
	1.00	39.0	21	18.1	10.6	8.9	---	#	33.3	12.3	11.5
.85	.50	10.2	6	7.0	8.7	6.3	15.6	9	11.1	13.3	9.3
	.75	10.2	6	8.3	9.5	6.8	20.9	12	13.8	14.3	9.7
	1.00	19.0	11	9.6	7.2	6.0	36.8	21	17.9	10.0	8.4
.90	.50	6.3	4	5.1	7.5	4.4	11.4	7	7.6	9.7	6.0
	.75	8.1	5	5.1	7.6	4.4	13.1	8	7.6	9.7	6.2
	1.00	9.6	6	5.1	7.0	5.1	18.1	11	10.1	10.5	6.9
.95	.50	4.5	3	3.6	6.7	3.9	7.7	5	4.7	6.7	4.6
	.75	4.5	3	3.6	6.6	3.8	7.7	5	4.7	6.7	4.6
	1.00	6.0	4	3.6	6.7	3.7	9.2	6	5.8	6.5	4.5

APPENDIX C

by N. Elliott and Y. S. Lin

University of Minnesota

Let $R_{k,m,n}$, $S_{k,m,n}$, $T_{k,m,n}$ (with all subscripts nonnegative), be as defined in (3.1) with $p_1 = 1 - q_1$, $p_2 = \theta$ and $p_3 = 1 - q_3$ as defined earlier. Using the recursive equations (3.2) and boundary conditions (3.3), we wish to show that the probability of a correct selection given by $A_{r,r,r}$ in (3.4) is a strictly increasing function of p_1.

Let v_N denote the vector (k,m,n) when the sum of the components $k + m + n$ equals N; let w_N denote another vector with component sum N'. We say that v_N majorizes $w_{N'}$ (written $v_N \succ w_{N'}$) if

(C 1) $$k' \geq k,\ m' \leq m,\ n' \leq n \text{ and } N' \leq N.$$

Let ψ denote a generic symbol for all three functions, R, S and T.

<u>Lemma 1</u>: If $v_N \succ w_{N'}$ then

(C 2) $$\psi_{k,m,n} \geq \psi_{k',m',n'}.$$

<u>Proof</u>. We need only consider $N > 3$ and k,m',n' all positive since the proof is trivial otherwise. To complete the induction proof, for example for $\psi = R$, we use (3.2) to write

(C 3) $$R_{k,m,n} - R_{k',m',n'} = p_1(T_{k-1,m,n} - T_{k'-1,m,n}) + q_1(S_{k,m-1,n} - S_{k',m'-1,n'})$$

and the result therefore holds for N if it holds for $N - 1$. A similar proof holds for $\psi = S$ and $\psi = T$.

<u>Lemma 2</u>: For all p_1, θ, p_3

(C 4) $$T_{k-1,m,n} \geq S_{k,m-1,n}$$

whenever all subscripts in (C 4) are nonnegative.

Proof. The result is trivial for $k = 1$ or $m = 1$ and we can therefore assume that $k \geq 2$ and $m \geq 2$.

If we start with the middle equation of (3.2) substitute for R from the first equation, and then use the second equation again to eliminate the new S, we eventually get $S_{k,m-1,n}$ as a function of T's only. This result can be written for all $m \geq 2$ as

$$\text{(C 5)} \quad S_{k,m-1,n} = (1 - \theta) \sum_{j=0}^{[\frac{m-2}{2}]} (\theta q_1)^j \, T_{k,m-1-2j,n-1} + \theta p_1 \sum_{j=0}^{[\frac{m-3}{2}]} (\theta q_1)^j \, T_{k-1,m-2-2j,n}],$$

where $[x]$ denotes the largest integer less than or equal to x. The sum of all the coefficients on the right side of (C 5) is at most one because the sum is exactly one in (3.2) and, at each step in deriving (C 5), we replaced a ψ-function by two terms with coefficients summing to one. Final terms of the form $\psi_{k,0,n} = 0$ are omitted in (C 5) and hence the sum of the coefficients on the right side of (C 5) must be at most one. Since $v_N = (k - 1, m, n)$ majorizes every vector on the right side of (C 5), it follows from lemma 1 that $T_{k-1,m,n}$ is greater than (or equal to) every T on the right side of (C 5). Using the fact that the sum of the coefficients in (C 5) is at most one, it follows from (C 5) that $T_{k-1,m,n} \geq S_{k,m-1,n}$; this proves lemma 2.

Let $\psi_{k,m,n}(MF)$ correspond to the more favorable configuration in which $p_1 = p + \epsilon$, $p_2 = 0$ and $p_3 = p$ for $\epsilon > 0$, and let $\psi_{k,m,n}(LF)$ correspond to the same with $\epsilon = 0$.

Theorem. For all nonnegative k, m, n

$$\text{(c 6)} \qquad \psi_{k,m,n}(MF) \geq \psi_{k,m,n}(LF).$$

Proof. We can assume k, m, n all positive and $N \geq 3$ since the proof is trivial otherwise. Using (3.2) and the inductive hypothesis

(C 7) $$S_{k,m,n}(MF) = \theta R_{k,m-1,n}(MF) + (1-\theta) T_{k,m,n-1}(MF)$$
$$\geq \theta R_{k,m-1,n}(LF) + (1-\theta) T_{k,m,n-1}(LF) = S_{k,m,n}(LF),$$

and a similar proof holds for $T_{k,m,n}$. For $R_{k,m,n}$ we obtain

(C 8) $$R_{k,m,n}(MF) = (p+\epsilon) T_{k-1,m,n}(MF) + (q-\epsilon) S_{k,m-1,n}(MF)$$
$$\geq p\, T_{k-1,m,n}(LF) + q\, S_{k,m-1,n}(LF) + \epsilon[T_{k-1,m,n}(MF) - S_{k,m-1,n}(MF)]$$
$$\geq R_{k,m,n}(LF),$$

since lemma 2 holds for all configurations; this proves the theorem.

Corollary. For procedure R_I with any r

(C 9) $$P\{CS|MF\} \geq P\{CS|LF\}.$$

Proof. Since the $P\{CS\}$ under procedure R_I is given by (3.4) the result (C 9) is an immediate consequence of the theorem above.

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